A computational comparison of methods diminishing spurious oscillations in finite element solutions of convection–diffusion equations

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We discuss the application of the finite element method to the numerical solution of the scalar convection–diffusion equation

$$-\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u = f \quad \text{in } \Omega, \qquad u = u_b \quad \text{on } \Gamma^D, \qquad \varepsilon\,\frac{\partial u}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma^N. \tag{1}$$

Here Ω is a bounded two–dimensional domain with a polygonal boundary $\partial\Omega$, Γ^D and Γ^N are disjoint and relatively open subsets of $\partial\Omega$ satisfying $\mathrm{meas}_1(\Gamma^D) > 0$ and $\overline{\Gamma^D \cup \Gamma^N} = \partial\Omega$, $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$, f is a given outer source of the unknown scalar quantity u, $\varepsilon > 0$ is the constant diffusivity, $\boldsymbol{b}$ is the flow velocity, and u_b, g are given functions.

Despite the apparent simplicity of problem (1), its numerical solution is by no means easy if convection is strongly dominant (i.e., if $\varepsilon \ll |\boldsymbol{b}|$). In this case, the solution of (1) typically possesses interior and boundary layers and hence the layers cannot be resolved properly. In particular, it is well known that the classical Galerkin finite element discretization of (1) is inappropriate in the convection–dominated regime since the discrete solution is typically globally polluted by spurious oscillations. Although, during the last three decades, an extensive research has been devoted to the development of methods which diminish spurious oscillations in the discrete solutions of (1), the numerical solution of (1) is still a challenge when convection strongly dominates diffusion.

One of the most efficient procedures is the streamline upwind/Petrov–Galerkin (SUPG) method developed by Brooks and Hughes [1] which is a higher–order method possessing good stability properties achieved by adding artificial diffusion in the streamline direction. Unfortunately, the SUPG method does not preclude spurious oscillations localized in narrow regions along sharp layers. Although these oscillations are usually small in magnitude, they are not permissible in many applications. Therefore, various terms introducing artificial crosswind diffusion in the neighborhood of layers have been proposed to be added to the SUPG formulation in order to obtain a method which is monotone or which at least reduces the local oscillations (cf. e.g. [2, 3] and the references there). This procedure is often referred to as discontinuity capturing (or shock capturing). The literature on discontinuity–capturing methods is rather extended and the various numerical tests published in the literature do not allow to draw a clear conclusion concerning their advantages and drawbacks. Therefore, our aim is to provide a review of discontinuity–capturing methods and to compare these methods computationally by means of several test problems.

References

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