On a Sandia Structural Mechanics Challenge Problem

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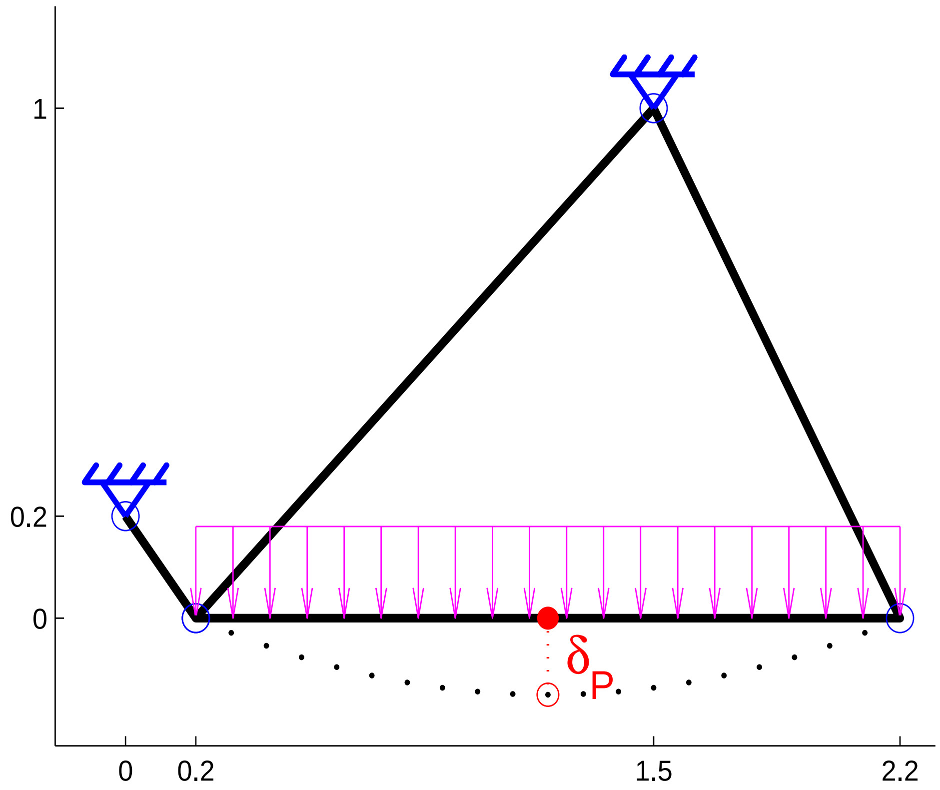
Prediction problem:

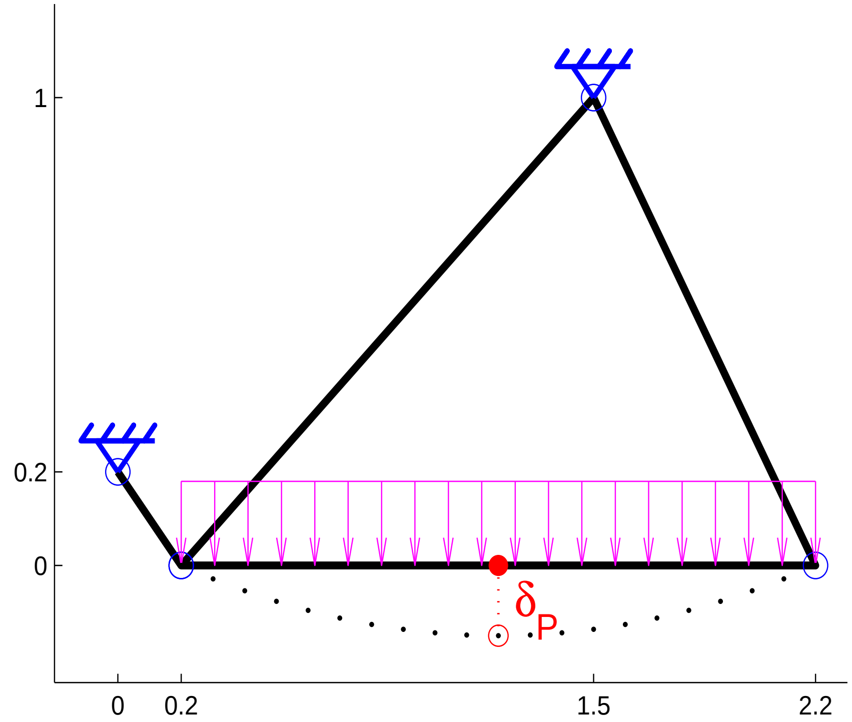
Does the displacement δ_P exceed a given limit (3 mm)?



We know:

- the geometry
- the load
- the mathematical model

We have limited information about E, the modulus of elasticity.



Information provided about E:

- five local values E_0
- five averaged values E_{20}

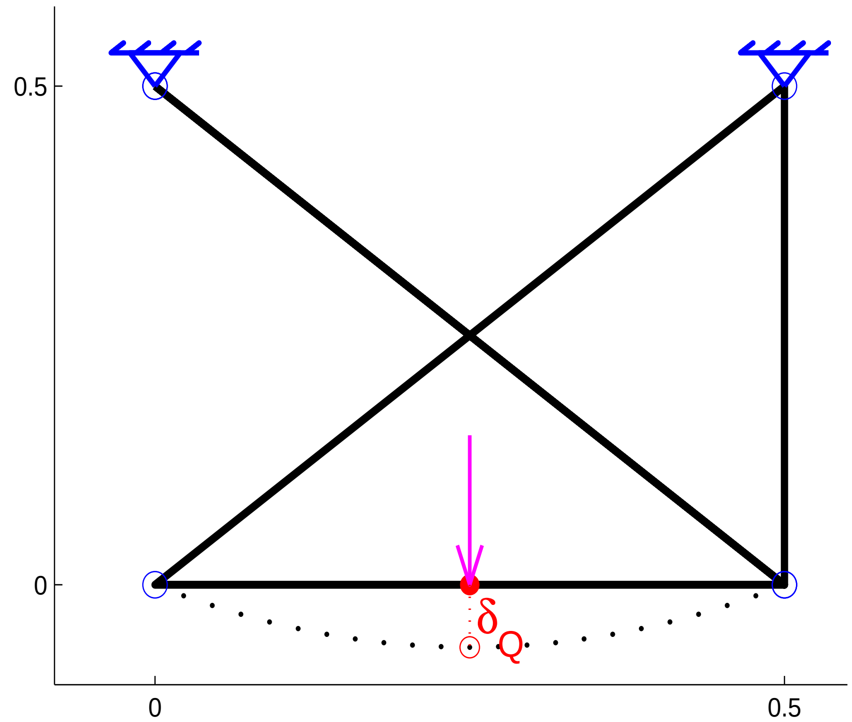
inferred from the elongation of sample rods 20 cm long

- two averaged values E_{80}

inferred from the elongation of sample rods 80 cm long

- δ_Q, a displacement of

a "similar" structure nicknamed the accreditation problem



The prediction problem was proposed by Ivo Babuška, Fabio Nobile, and Raul Tempone as one of the uncertain input data problems to challenge the participants of Validation Challenge Workshop, Sandia National Laboratories, Albuquerque, NM, USA, May 21-23, 2006.

In the problem, three levels of information (i.e., sets of measurements) are offered. We use the poorest set.

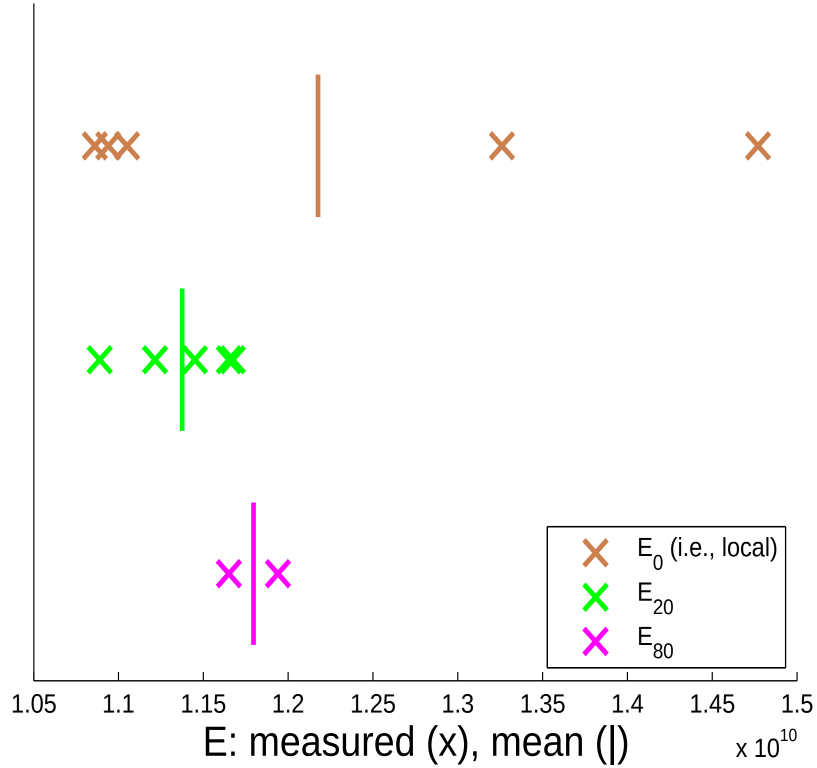
Difficulties:

- insufficient number

of experiments

- uncertain probability distribution
- poor estimates

of probability-related parameters



Stochastic process

$E \equiv E(x, \omega)$ is a stationary random field (w.r.t. x); $x \in [0, L_i]$ where L_i is the length of the i-th rod. For some purposes, $E(x, \omega)$ can be reduced to $E_0(\omega)$, the field independent of x.

The expected value of E (the mean): $E_{\mathrm{m}} = \mathbb{E}(E_0(\omega)) =$ constant independent of x and i.

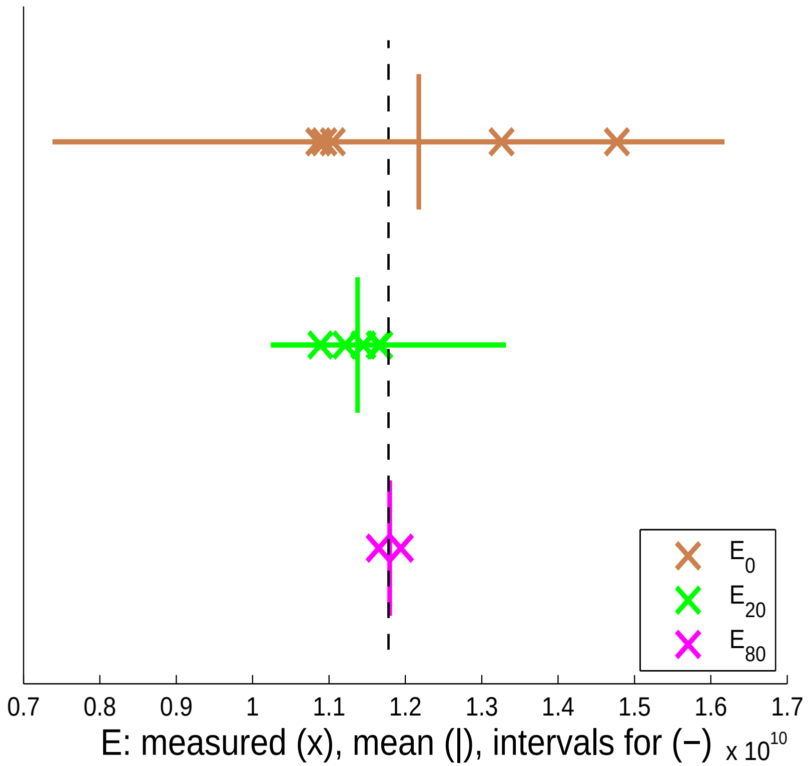
We have to assume that $E(x, \omega)$ and $E(y, \omega)$ are not independent – especially if x is "close" to y.

Idea:

- Choose intervals I_0 and I_{20} such that they contain the measured values of E_0 and E_{20}, respectively.
- Assume a probability distribution of E_0.
- For $1/E$, assume a covariance function with an unknown correlation length L_{corr}.
- Calculate the correlation length L_{corr}.
- By knowing L_{corr}, infer an interval for E_{80} and check it against the measured values of E_{80}.
- Infer an interval for δ_Q and check it against the value of δ_Q coming from the accreditation test.
- Infer an interval for δ_P and check it against the 3 mm limit given in the prediction problem. Try to make a conclusion.

Much space for expert opinion!

Remark: B.&N.&T. call the measured E_{20} values *calibration data* and the measured E_{80} values *validation data*.



The intervals are constructed around a central value (dashed) and interpreted as either the respective intervals in which both E_0 and E_{20} are uniformly distributed or the intervals covering 95% of normally distributed values E_0 and E_{20}.

Let us recall that $E_0(\omega)$ is a random field of local values of E, that is, a field identical to $E(x,\omega)$ except for the localization at a particular x.

We assume

$$\mathrm{cov}\left(\frac{1}{E}\right) = \mathbb{E}\left[\frac{1}{E(x,\omega)}\frac{1}{E(y,\omega)}\right] - \left(\mathbb{E}\left[\frac{1}{E_0(\omega)}\right]\right)^2$$
$$= var\left(\frac{1}{E_0}\right) g(x,y,L_{\mathrm{corr}}),$$

where

$$g(x,y,L_{\mathrm{corr}}) = \exp\left(-\frac{|x-y|}{L_{\mathrm{corr}}}\right).$$

Other choices of g are possible. Take $g(x,y,L_{\mathrm{corr}}) = \exp\left(-\frac{|x-y|^2}{L_{\mathrm{corr}}^2}\right)$, for instance.

Equation for L_{corr}

If a uniform rod of length L and cross section area A is axially loaded by a force F, then δ_L, its elongation, is a random variable:

$$\delta_L(\omega) = \frac{F}{A} \int_0^L \frac{1}{E(x,\omega)}\, dx.$$

Then

$$\begin{aligned}
var(\delta_L) &= \mathbb{E}\left[\delta_L^2\right] - (\mathbb{E}[\delta_L])^2 = \ldots \text{ after some algebra } \ldots \\
&= \frac{F^2}{A^2} \int_0^L \int_0^L \mathbb{E}\left[\frac{1}{E(x,\omega)} \frac{1}{E(y,\omega)}\right] - \left(\mathbb{E}\left[\frac{1}{E_0(\omega)}\right]\right)^2 dx\, dy \\
&= \frac{F^2}{A^2} \int_0^L \int_0^L \text{cov}\left(\frac{1}{E}\right) dx\, dy \\
&= \frac{F^2}{A^2}\, var\left(\frac{1}{E_0}\right) \int_0^L \int_0^L \exp\left(-\frac{|x-y|}{L_{\text{corr}}}\right) dx\, dy.
\end{aligned}$$

However, if we define E_L as the effective modulus of elasticity inferred from the prolongation of the rod of length L, we obtain

$$\delta_L(\omega) = \frac{FL}{A}\frac{1}{E_L(\omega)}$$
$$var(\delta_L) = \frac{F^2L^2}{A^2}\, var\Big(\frac{1}{E_L}\Big).$$

By comparing both equations, we eliminate $var(\delta_L)$ and arrive at

$$\frac{var(1/E_L)}{var(1/E_0)} = \frac{1}{L^2}\int_0^2\int_0^2 \exp\Big(-\frac{|x-y|}{L_{\mathsf{corr}}}\Big)\, dx\, dy. \qquad (1)$$

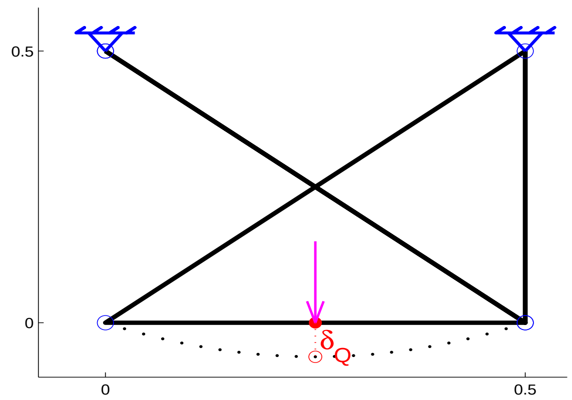
To solve (1), we evaluate $var(1/E_0)$ by means of the assumed probability distribution of E_0 in the interval I_0. We evaluate $var(1/E_L)$, where $L = 20$ cm, in a similar way using I_{20}. After exact integration of the r.h.s. of (1) (done by `Maple`), the r.h.s. becomes a function of L_{corr}, and (1) can be solved numerically for L_{corr}.

As soon as $var(1/E_0)$ is fixed by assumption and L_{corr} is known from (1), we can use (1) to directly calculate $var(1/E_L)$ for $L = 80$ cm and other lengths. We assume that $var(1/E_{80})$ corresponds to either a uniform or normal distribution of E_{80}. Under these assumptions, we can infer I_{80} and check whether or not the validation data lie in I_{80}.

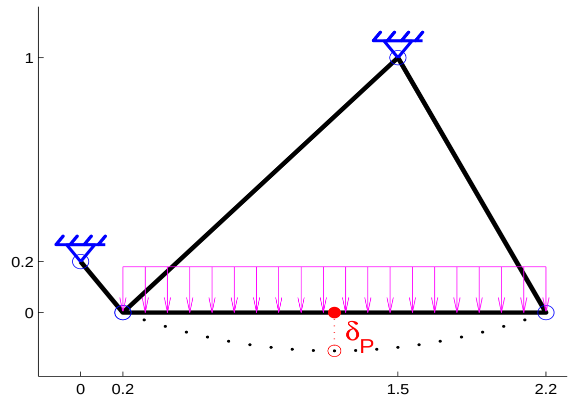
In a similar way but with much less effort, we can infer

$$\mathbb{E}[\delta_L] = \frac{FL}{A}\mathbb{E}\Big[\frac{1}{E_0}\Big].$$

Since δ_Q, see the accreditation problem, can be expressed as a linear combination of δ_{L_i}, $i = 1, 2, 3, 4$, the same technique enables us to obtain the mean value of δ_Q and $var(\delta_Q)$.



Similarly, we infer the mean value of δ_P and $var(\delta_P)$, the quantities important for addressing the prediction problem.

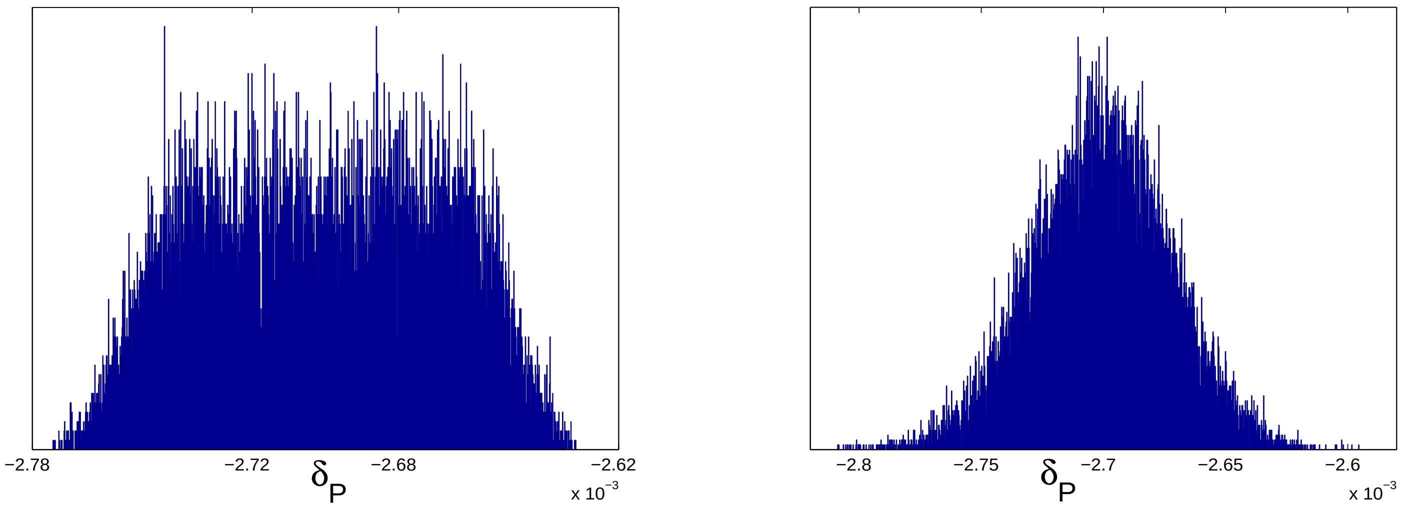


Remark: The bending of transversaly loaded beams is expressed through the Green function. To compute the corresponding variation of the vertical displacements, integrals such as

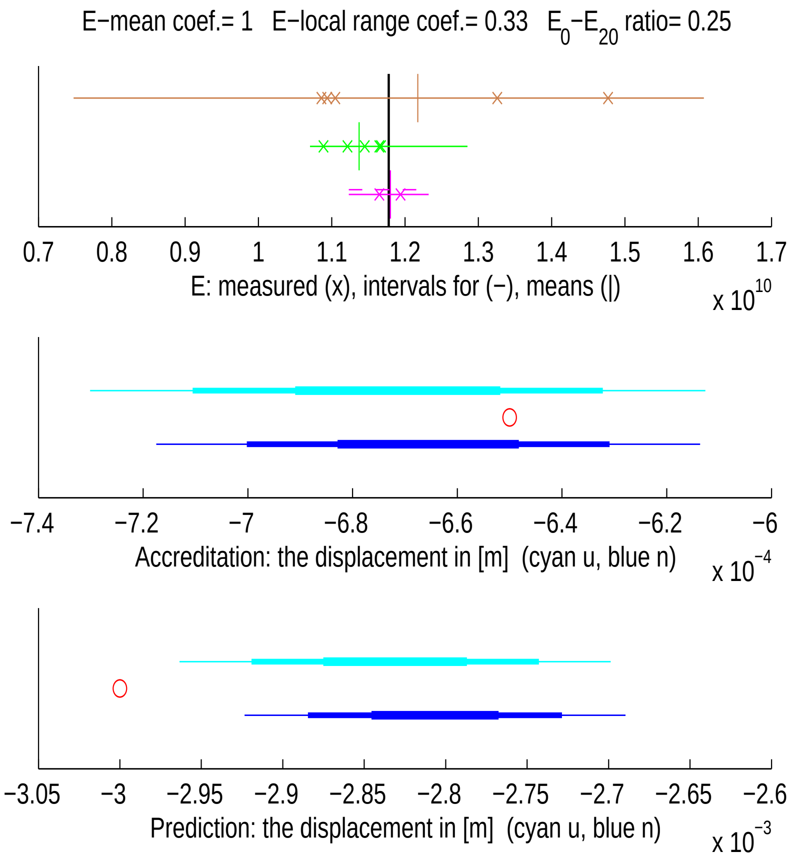
$$\int_0^L \int_0^L \phi(x)\psi(y)\,\mathsf{exp}\left(-\frac{|x-y|}{L_{\mathsf{corr}}}\right) dx\,dy, \qquad (2)$$

have to be evaluated. In (2), the product $\phi\psi$ is a continuous piecewise quadratic or cubic function. Again, `Maple` is able to analytically integrate expression (2) and convert the resulting formulae into `Matlab` code.

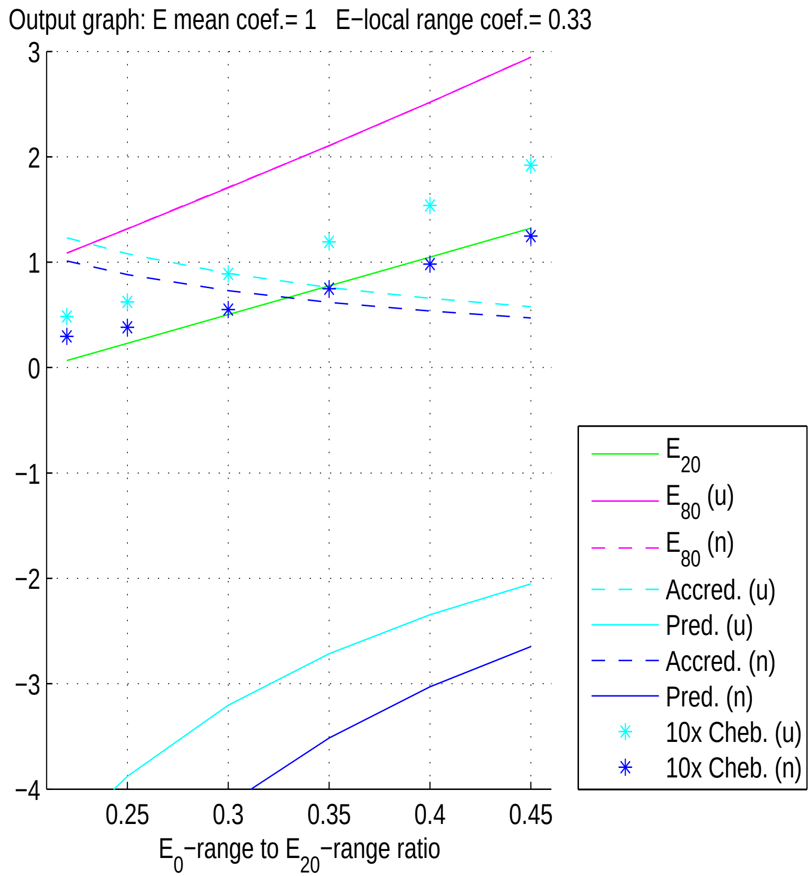
To get an insight, it also helps to apply the Monte Carlo method to both structures, though it simulates partly different mathematical model. Indeed, sample structures are generated with effective elasticity moduli (E_{L_i}) that are used even in the beam loaded by the transversal force; this is not exactly the model that we have studied.



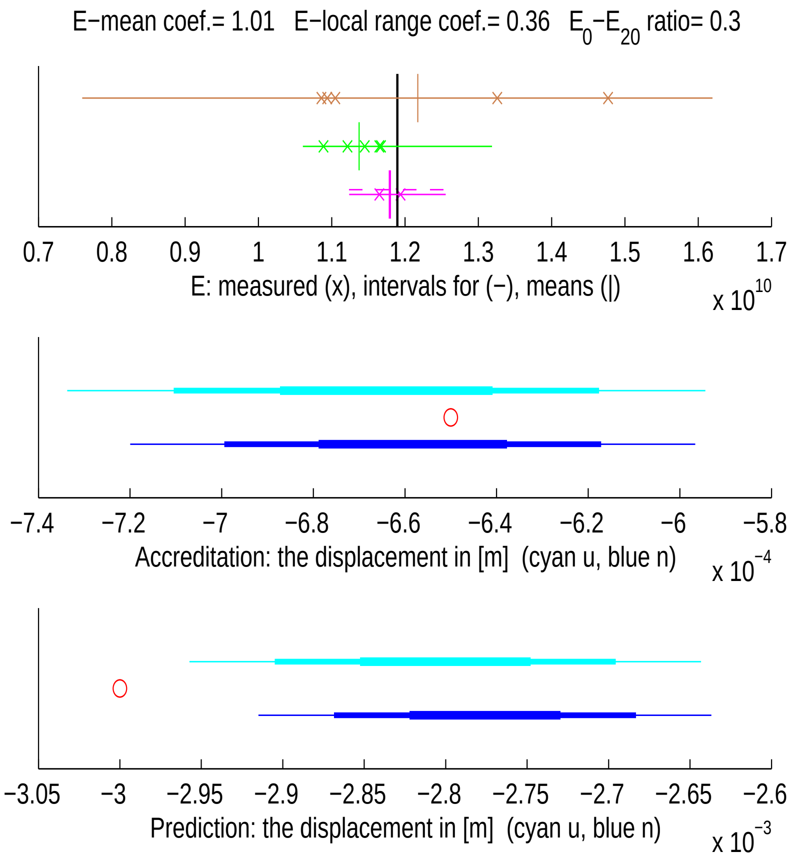
The results of our calculations will be presented in graphs that, we believe, can help the analyst to get some insight into the response of the prediction problem to the assumptions made.



E−mean coef.= 1 E−local range coef.= 0.33 E0−E20 ratio= 0.25
0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7
E: measured (x), intervals for (−), means (|)
x 10^10
−7.4 −7.2 −7 −6.8 −6.6 −6.4 −6.2 −6
Accreditation: the displacement in [m] (cyan u, blue n)
x 10^-4
−3.05 −3 −2.95 −2.9 −2.85 −2.8 −2.75 −2.7 −2.65 −2.6
Prediction: the displacement in [m] (cyan u, blue n)
x 10^-3



Output graph: E mean coef.= 1 E−local range coef.= 0.33
3 2 1 0 −1 −2 −3 −4
0.25 0.3 0.35 0.4 0.45
E0−range to E20−range ratio
E20
E80 (u)
E80 (n)
Accred. (u)
Pred. (u)
Accred. (n)
Pred. (n)
10x Cheb. (u)
10x Cheb. (n)


E−mean coef.= 1.01 E−local range coef.= 0.36 E0−E20 ratio= 0.3
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
E: measured (x), intervals for (−), means (|)
x 10^10
−7.4
−7.2
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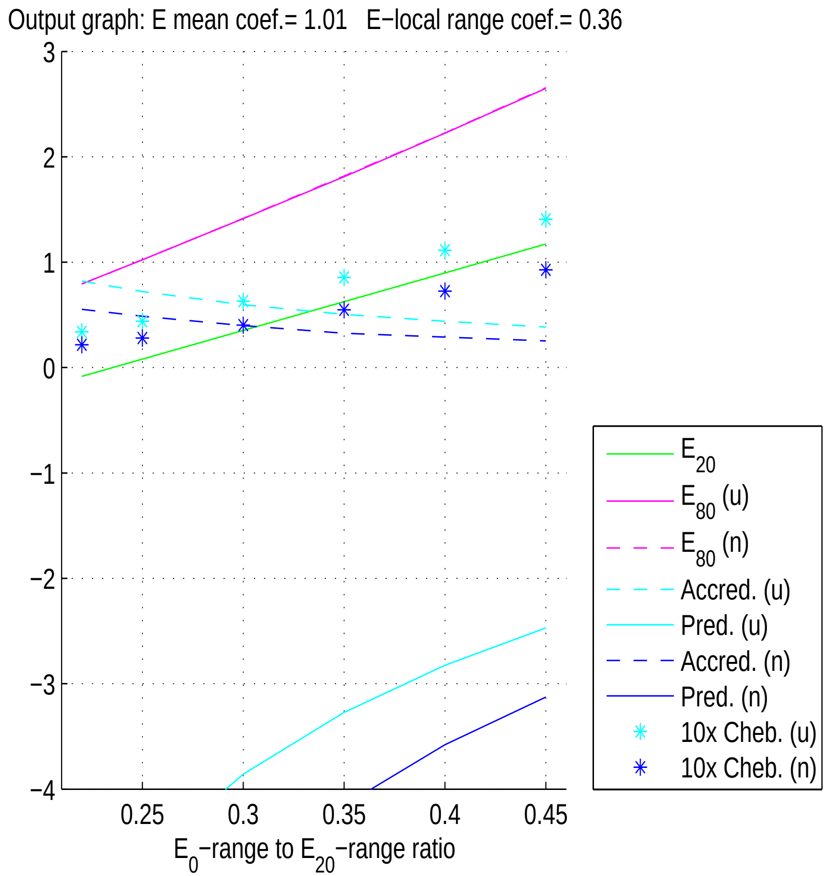
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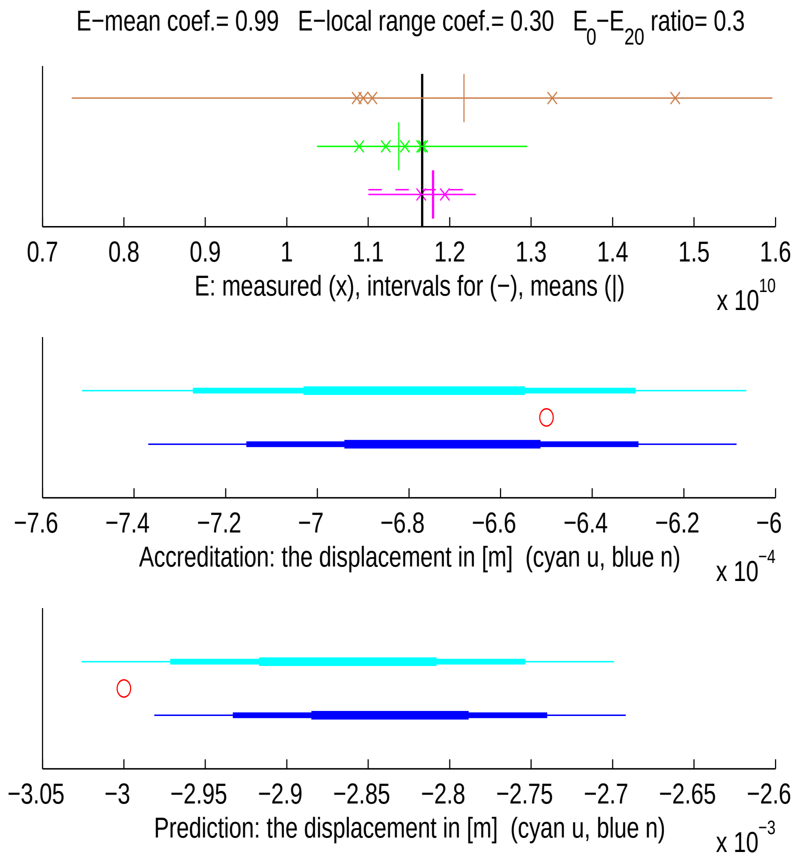
E−mean coef.= 0.99 E−local range coef.= 0.30 E0−E20 ratio= 0.3
0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6
E: measured (x), intervals for (−), means (|)
x 10^10
−7.6 −7.4 −7.2 −7 −6.8 −6.6 −6.4 −6.2 −6
Accreditation: the displacement in [m] (cyan u, blue n)
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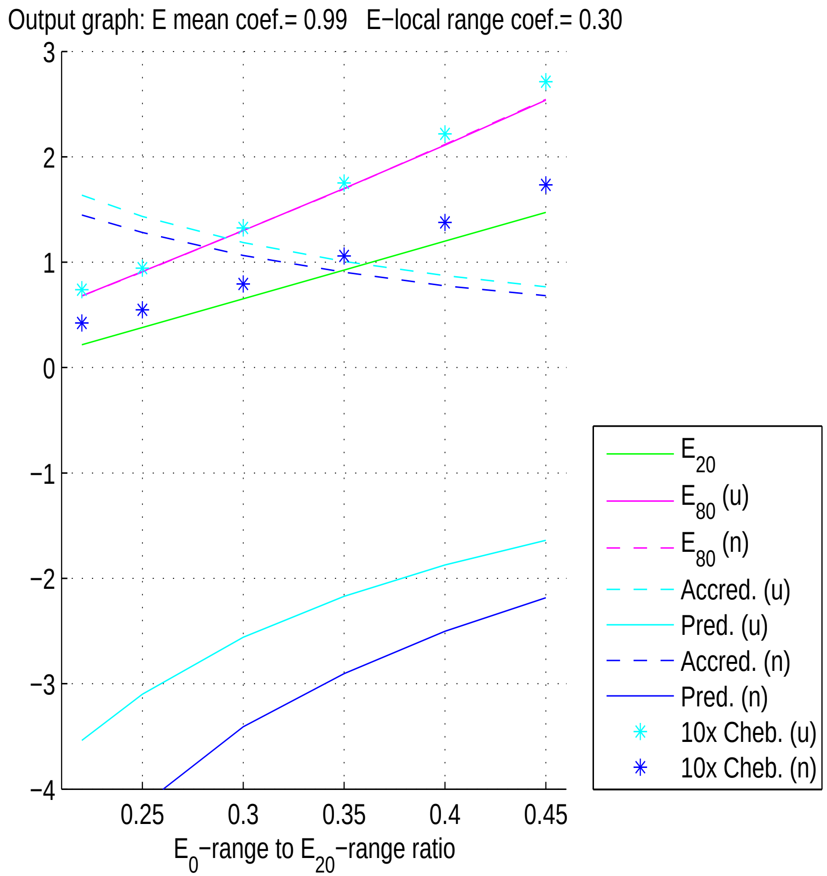
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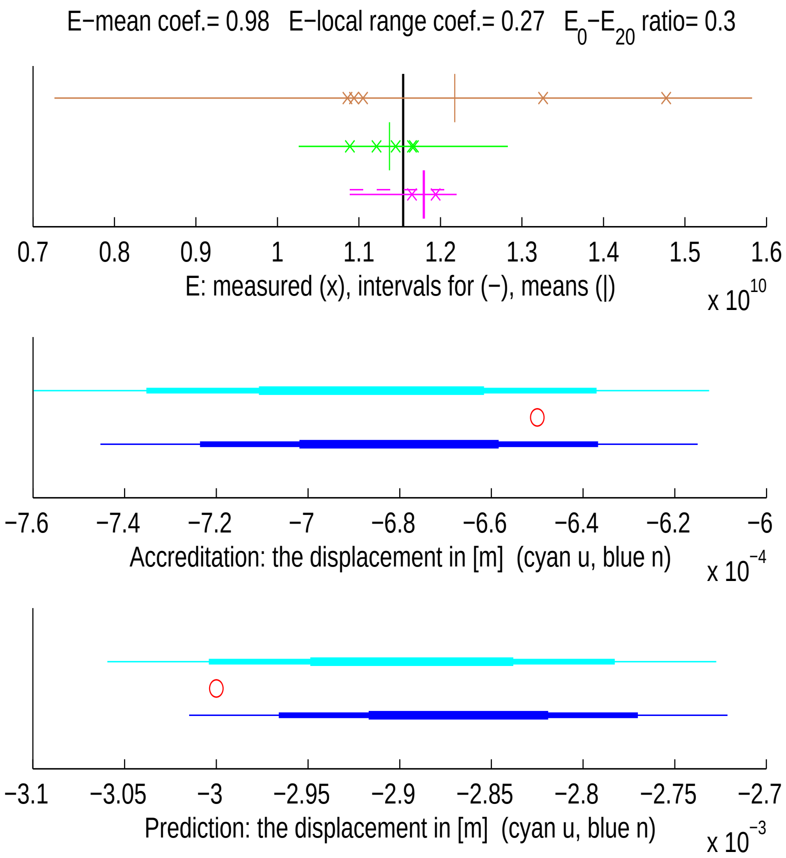
E−mean coef.= 0.98 E−local range coef.= 0.27 E0−E20 ratio= 0.3
0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6
E: measured (x), intervals for (−), means (|)
x 10^10
−7.6 −7.4 −7.2 −7 −6.8 −6.6 −6.4 −6.2 −6
Accreditation: the displacement in [m] (cyan u, blue n)
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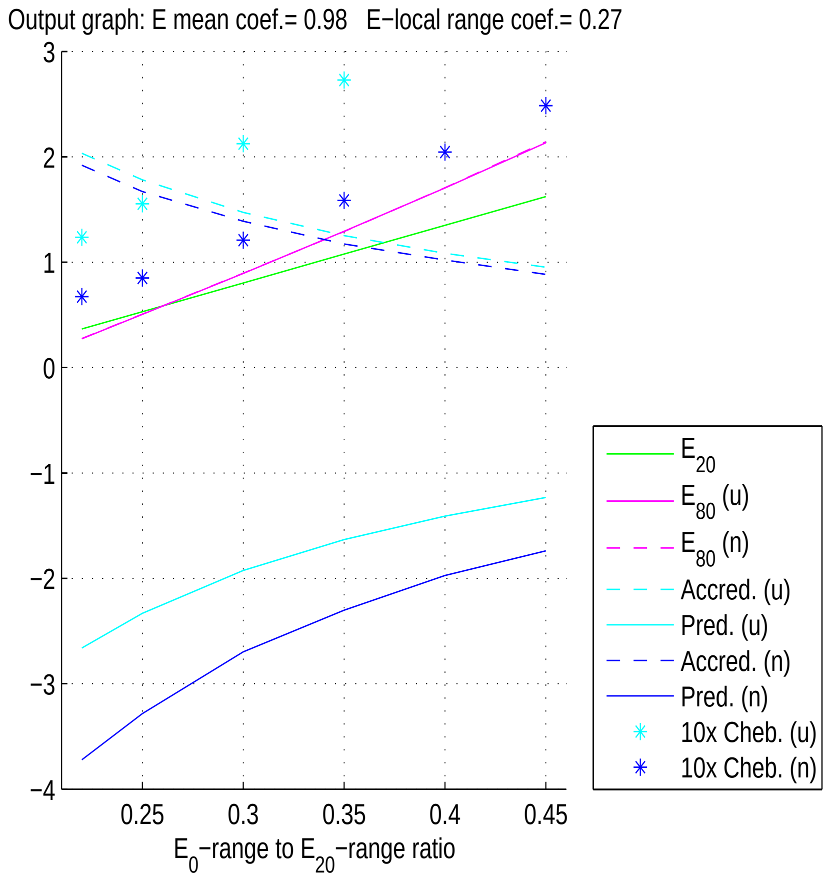
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E0−range to E20−range ratio
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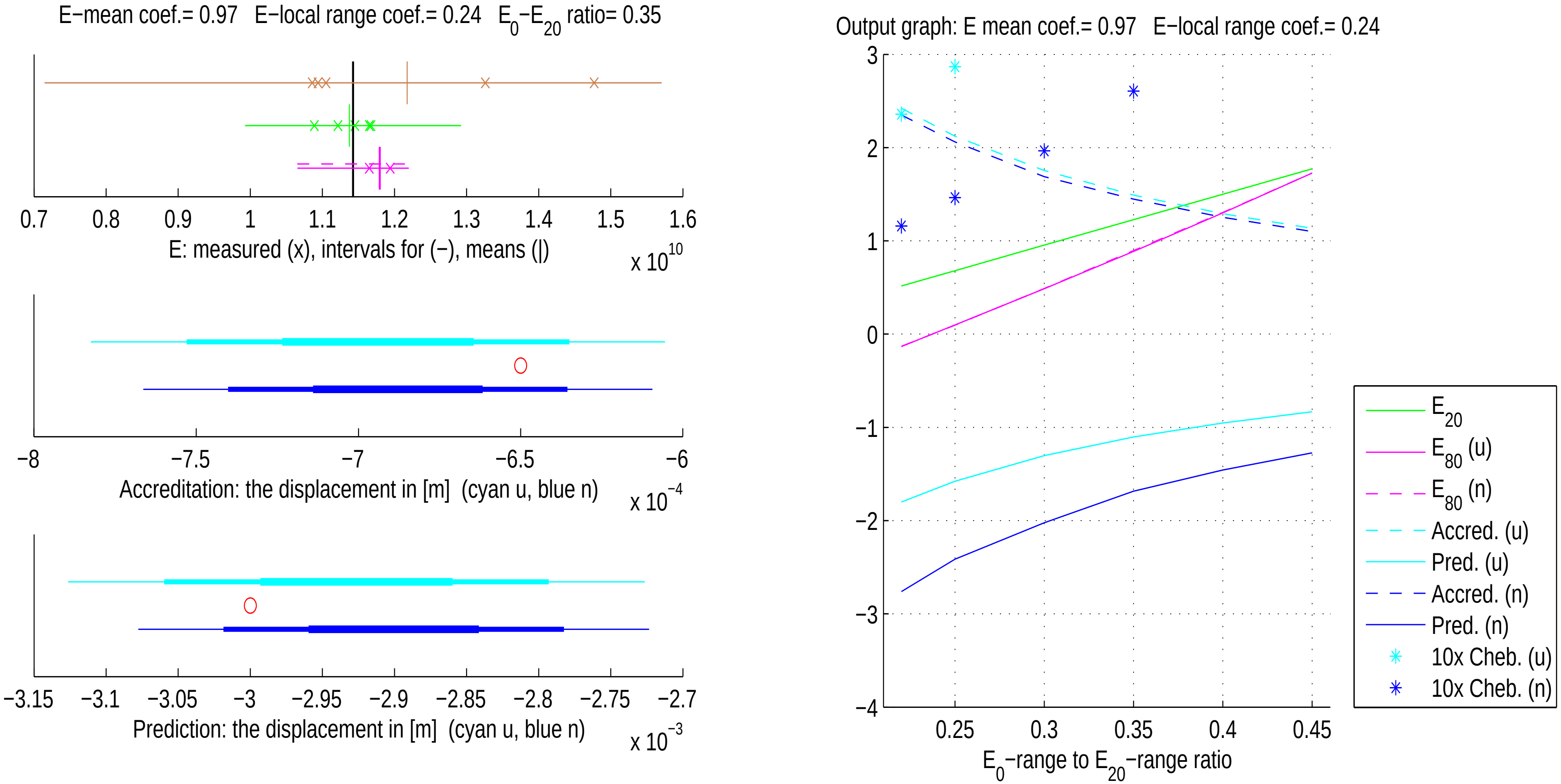
E−mean coef.= 0.97   E−local range coef.= 0.24   E0−E20 ratio= 0.35
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E80 (n)
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Pred. (u)
Accred. (n)
Pred. (n)
10x Cheb. (u)
10x Cheb. (n)

Observations

- The most important quantity is $E_{\mathrm{m}} = \mathbb{E}[E_0]$. The smaller E_{m}, the greater probability that δ_P exceeds the limit.
- If E_{m} becomes too small, say less than $0.98 E_{\mathrm{M}}$, where E_{M} is the mean of measured E, then it is harder to comply with the validation and accreditation data.
- The greater E_{m} (above E_{M}), the smaller δ_{P} (good news). However, it is harder to comply with the calibration tests. Moreover the calibration dataset then becomes more and more "one-sided", which is less and less probable.
- Although, at the first glance, the predictions based on E uniformly distributed seems to be worse (closer to 3 mm) than the predictions based on the Gaussian distribution of E, they are not that much different because the uniform distribution leads to short "tails", see the histograms of δ_P.

Answers to dilemma Yes ($\delta_P \geq 3$) No ($\delta_P < 3$)

1) If you follow the rule that a model both strict and fitting to the data is the best choice, say No. You will sleep like an innocent baby.
2) If you are a realist but if you believe that things mostly end in good, say No. You will sleep well.
3) If you know the harsh side of life, say No. You will feel that you have more than a fifty-fifty chance to be right.
4) If it is a matter of life and death, say No. Simply try to believe in my sixth sense.
5) If you do not feel any inclination to *yes* or *no* and if you do not hear an inner voice, quit problems with uncertain data.

I thank Ivo Babuška, Fabio Nobile, and Raul Tempone for inventing this puzzle and for giving me lectures on variance, covariance, and loaded beams.